Building and Classifying Doppelgangers of Unlabeled Posets

Thomas Browning, Max Hopkins, Zander Kelley

June 2016

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1 Introduction

Unlabeled partially ordered sets may be partitioned into equivalence classes by their order polynomials. We call two posets "doppelgangers" if they lie within the same equivalence class [1]. The order polynomial closely relates to the multivariate generating function on labeled posets [3], i.e. its value at m for a poset P, $F_P(m)$, is given by the generating function evaluated at $(\underbrace{1,...,1}_{m},0,...)$.

Because doppelgangers are a weakening of the multivariate generating function, doppelganger properties carry over to multivariate equivalence, for instance the height invariant. However, some properties enjoyed by the multivariate generating function do not apply to doppelgangers, such as the width invariant. For an initial discussion of the order polynomial, see Stanley's "Enumerative Combinatorics", for doppelgangers see Hamaker's "Doppelgängers: Bijections of Plane Partitions."

We begin our paper by introducing a new operation, a generalization of standard poset operations, which provides a way to build doppelgangers of any arbitrary poset P through decomposition. While this decomposition is interesting in its own right, its use for the creation of doppelgangers requires further knowledge, which we gain through a recurrence relation on the order polynomial, inclusion-exclusion on incomparable elements of a poset. This method mirrors inclusion-exclusion on the order polytope, where coefficients of the decomposition can be found in hyper-plane arrangements (Feray). Note that any previous knowledge of doppelgangers may be applied to the new decomposition, even noting that any poset is a doppelganger with its dual provides new insight. Using only a single step of this inclusion-exclusion, we provide an elegant proof of the poset reciprocity theorem (Stanley), which in turn proves a height invariant on doppelganger classes. Using four calculable invariants, we classify posets of height $|P|$, $|P|-1$, and $|P|-2$, and offer some new directions for possible research.

2 Definitions and Notation

A P-partition of height $m \in \mathbb{N}$ of a poset P is an order preserving map $F\colon P \to [m]$. The set of such maps is denoted by $\mathscr{P}\mathscr{P}^{[m]}(P)$ and its cardinality is denoted by $F_P(m) = |\mathscr{P}\mathscr{P}^{[m]}(P)|$. Furthermore, $F_P(m)$ is known to be a polynomial in m of order $|P|$, and is thus called the order polynomial. Posets P and Q are doppelgangers if their order polynomials are equivalent, and we write $P \sim Q$. The dual of a poset P, P^*, is the poset where all relations are flipped, it is obvious that $P \sim P^*$.

We introduce some notation for convenience used throughout the paper. We will write chains of height k as C_k, and the anti-chain of height k as A_k. Note that the poset with a single element could then either be denoted as C_1 or A_1, we pick the convention of C_1 in this paper. Let the height of P be $h(P)$, let the width of P be $w(P)$, let the number of linear extensions of P be $e(P)$. We use $\mathbb{N} = \{1, 2, \cdots, \}$ to denote the natural numbers and we use the notation $[m] = \{1, \cdots, m\}$.

3 Ur-Decomposition

In this section, we present an operation on posets which generalizes three standard operations: disjoint unition, ordinal sum, and ordinal product.

Definition 1. *For a poset $\mathscr{P} = \{x_1, \cdots, x_n\}$ and a sequence of posets $\{P_1, \cdots, P_n\}$, let $\mathscr{P}[x_k \to P_k]_{k=1}^n$ be the poset on $\bigcup_k P_k$ with the following operation:*

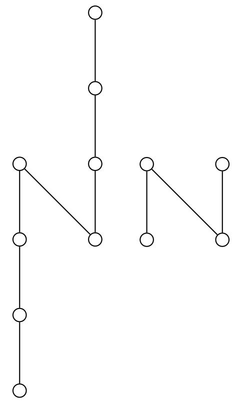
$$\text{For } p \in P_j, q \in P_k,\ p \leq q \text{ when } \begin{cases} p \leq q & j = k \\ x_j \leq x_k & j \neq k \end{cases}.$$

We denote this as the Ur-operation on $\mathscr{P}$ by $\{P_1, \cdots, P_n\}$. All P_k are assumed to be C_1 if not specified.

Note that the disjoint sum operation denoted by $P_1 + P_2$ can be expressed as $A_2[x_k \to P_k]_{k=1}^2$, the ordinal sum operation denoted by $P_1 \oplus P_2$ can be expressed as $C_2[x_k \to P_k]_{k=1}^2$, and the ordinal product operation denoted by $P \otimes Q$ can be expressed as $P[x_k \to Q]_{k=1}^n$. In these ways, the Ur-operation generalized many of the standard operations on posets. Recall that a poset P, $|P| > 1$, is called prime if it cannot be expressed as the ordinal sum or disjoint union of two posets. Then these two operations uniquely decompose (up to commutivity of $+$) all posets into primes and posets of one element. This decomposition is known as the series-parallel decomposition. We may generalize this definition by saying a poset P, $|P| > 2$, is a strong prime if it cannot be expressed as a result of a non-trivial Ur-Operation. Note that all strong primes are primes but not vice versa. Having introduced this new operation, we wish in some way to motivate it. The following lemma describes how the order polynomial of $\mathscr{P}[x \to P]$. Not only is this enough to lead to results on dopplegangers, but giving providing equation in full generality (multiple points blown up) is complicated enough to obscure its use.

Lemma 2. *For a poset $\mathscr{P}$ with $x \in \mathscr{P}$, a poset P, and $m \in \mathbb{Z}^+$,*

$$F_{\mathscr{P}[x \to P]}(m) = \sum_{f \in S} F_P \left(1 + \min_{x \leq y} f(y) - \max_{y \leq x} f(y)\right)$$



(a) (b)

Figure 1: A prime and its corresponding strong prime

where S is the set of order preserving functions $f\colon (\mathscr{P}-x)\to[m]$, $\min_{x\le y} f(y)=m$ when there does not exist a y such that $x\le y$, and $\max_{y\le x} f(y)=1$ when there does not exist a y such that $y\le x$.

Proof. To count the number of order preserving maps $f\colon \mathscr{P}[x\to P]\to[m]$, we may first iterate over all valid assignments of f on $\mathscr{P}-x$ and for each one count the number of assignments of f on P that are compatible with our initial choice of f on $\mathscr{P}-x$. The set of valid assignments of f on $\mathscr{P}-x$ is precisely the set S. Then with the values of f assigned on $\mathscr{P}-x$, the only restriction of f on P are that $f(y)\le f(P)$ for all $y\le x$ or $1\le f(P)$ if no such y exist, and that $f(P)\le f(y)$ for all $x\le y$ or $f(P)\le m$ if no such y exists. This is equivalent to the condition that f maps P to the interval $\left[\max_{y\le x} f(y), \min_{x\le y} f(y)\right]$. The number of ways to do so is precisely $F_P\left(1+\min_{x\le y} f(y)-\max_{y\le x} f(y)\right)$. □

While this lemma only covers replacing a single element in $\mathscr{P}$, full generality of the lemma simply follows from its repeated application. Moreover, the lemma alone is enough to motivate our use of the Ur-Operation by the following corollary:

Corollary 3. *For a poset $\mathscr{P}$ with $x\in\mathscr{P}$ and posets $P\sim Q$, $F_{\mathscr{P}[x\to P]}=F_{\mathscr{P}[x\to Q]}$.*

Repeated applications of which provides our first important result:

Theorem 4. *For a poset $\mathscr{P}$ with $\{x_1,\cdots,x_n\}\subseteq\mathscr{P}$ and two sequences of posets $\{P_1,\cdots,P_n\}$ and $\{Q_1,\cdots,Q_n\}$ such that $P_i\sim Q_i$, $\mathscr{P}[x_k\to P_k]_{k=1}^n\sim\mathscr{P}[x_k\to Q_k]_{k=1}^n$.*

Proof. We shall proceed by induction on n. The base case of $n=1$ is corollary 1. Now suppose that the result holds for all $n\le N$ and that $n=N+1$. By our inductive assumption $\mathscr{P}[x_k\to P_k]_{k=1}^N\sim\mathscr{P}[x_k\to Q_k]_{k=1}^N$. Then we know there exists a bijection between the plane partitions of $P=\mathscr{P}[x_k\to P_k]_{k=1}^N$ and $Q=\mathscr{P}[x_k\to Q_k]_{k=1}^N$, but we need to prove there exists a difference preserving bijection g such that

$$\min_{x\le y} f(y)-\max_{y\le x} f(y)=\min_{x\le y} g(f)(y)-\max_{y\le x} g(f)(y)$$

Note that $|P|=|Q|=m$, and can be written as $\mathscr{P}[x_k\to P'_k]_{k=1}^m$ and $\mathscr{P}[x_k\to Q'_k]_{k=1}^m$ respectively, where for $k<n$ $P'_k=P_k$ and $Q'_k=Q_k$, and for $k>n$, $P'_k=Q'_k=C_1$. Then P and Q are partitioned by P_k and Q_k such that $P_k\sim Q_k$. Furthermore, note that if $P_k\sim Q_k$, then the number of plane partitions on P_k with a specified maximum and minimum value is the same as on Q_k (This is an inclusion-exclusion argument on values of the order polynomial). Lastly, note that the relations between the P_ks and the Q_ks are the same, namely those given by $\mathscr{P}$. Consider an assignment of min/max values to each k (except on P_{N+1}) s.t. the relations on $\mathscr{P}$ are satisfied; there are the same number of plane partitions with such an assignment for P and Q because $\mathscr{P}$ is common to both. But it is clear that the difference at x_{N+1} will be the same for partitions with the same min/max values, and there must then exist a difference preserving bijection. This gives $P[x_{N+1} \to P_{N+1}] \sim Q[x_{N+1} \to Q_{N+1}]$ or equivalently

$$\mathscr{P}[x_k \to P_k]_{k=1}^{N+1} \sim \mathscr{P}[x_k \to Q_k]_{k=1}^{N+1}$$

□

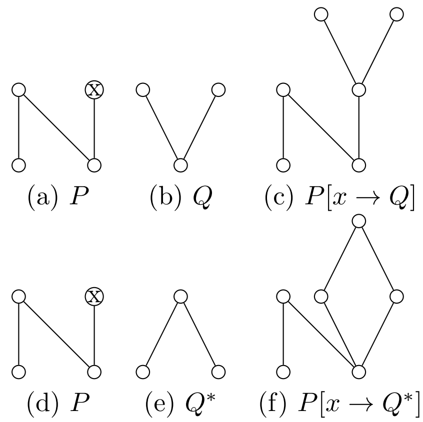


Figure 2: An example of dopplegangers due to the Ur-operation.

Theorem 1 allows us to build new doppelgangers out of the old and arbitrary posets. This can be applied to currently understood doppelgangers as well as those we present later in this paper through commutativity of ordinal sum and chain decomposition. However, the order polynomial is still calculationally too difficult to find, lemma 1 requires summing over all plane partitions of $\mathscr{P} - \{x\}$, which is $\#P$ hard. However, lemma 1 can be used as well to calculate simple order polynomials. For instance, it becomes clear how ordinal sum and disjoint union act on the order polynomial.

Proposition 5. *$F_{P+Q}(m) = F_P(m)F_Q(m)$, and $F_{P\oplus Q} = \sum_{i=1}^{m} F_Q(1+m-i)(F_P(i) - F_P(i-1))$ where $F_P(0)$ is defined to be 0.*

Proof.

$$\begin{aligned} F_{P+Q} &= F_{(P+e)[e\to Q]} \\ &= \sum_{f\in P} F_Q(m) \\ &= F_P(m)F_Q(m) \\ F_{P\oplus Q} &= F_{(P\oplus e)[e\to Q]} \\ &= \sum_{f\in P} F_Q\left(1+m-\max_{y\leq x} f(y)\right) \\ &= \sum_{i=1}^{m} F_Q(1+m-i)\left(F_P(i) - F_P(i-1)\right) \end{aligned}$$

Where the last step follows by re-indexing by the maximum of each plane partition. □

Moving away from the uses of lemma 1 and the special case operations, we seek a unique decomposition which illuminates the results of theorem 1, as well as which generalizes the standard series-parallel decomposition. However, the proof of this decomposition requires slightly more machinery than currently presented. To begin, we consider when some poset P could have been created via the Ur-Operation. This can only be the case if we can find some subposet which is reducible to a point, more formally

Definition 6. *A subset of a poset $\{x_k\} \subset P$ is reducible to a point (an RAP) when for every $y \in P - \{x_k\}$, either $y \leq \{x_k\}$, $\{x_k\} \leq y$, or $\{x_k\}$ and y are incomparable. An RAP $\{x_k\}$ of P is maximal when it is neither P nor a subset of any other RAPs other than P.*

Notably, an Ur-operation is an expression of the form $\mathscr{P}[x_k \to P_k]_{k=1}^n$ where each P_k is an RAP. RAPs should be considerd under two circumstances. If there does not exist Non-trivially intersecting RAPs such that their union is the entire poset, then RAPs are closed under union and intersection. This allows for the result that in this non-degenerate case, RAPs partition the poset. The degenerate case corresponds to when a poset can be expressed as an ordinal sum or disjoint union. Then RAPs may be used to further decompose prime posets, and give a more geometric description of strongly prime posets.

Proposition 7. *Any poset P, $|P| > 2$ is a strong prime if and only if it does not contain any non-trivial RAPs.*

Proof. We will show that a poset is not a strong prime if and only if it contains an RAP. For the forward direction, suppose that P is not a strong prime. By definition, $P = \mathscr{P}[x_k \to P_k]$. Then P_1 is an RAP of P. For the reverse direction, suppose that L is an RAP of P. Let $\mathscr{P}$ be the poset given by reducing L to a point x. Then $P = \mathscr{P}[x \to L]$ and P is not a strong prime. □

Lemma 8. *If x and y are RAPs of P with $x \cap y \neq \varnothing$, then $x \cap y$ and $x \cup y$ are RAPs of P.*

Proof. Suppose that $z \in P - x \cup y$. Then xRz and yQz for some relations R, Q. Since $x \cap y \neq \varnothing$, $R = Q$ and $(x \cup y)Rz$ which shows that $x \cup y$ is an RAP. Since $x \cap y \subseteq x$, $x \cap y$ is RAP to $P - x$. Since $x \cap y \subseteq y$, $x \cap y$ is RAP to $P - y$. Then $x \cap y$ is RAP to $(P - x) \cup (P - y) = P - (x \cap y)$. □

Proposition 9. *For any prime poset P, the maximal RAPs of P partition P.*

Proof. Since every point is in an RAP, every point is in a maximal RAP. Suppose that x and y are two maximal RAPs with $x \cap y \neq \varnothing$. If $x \cup y \neq P$ then by Lemma 2 $x \cup y$ would be an RAP which contradicts the maximality of x and y. Then $x \cup y = P$ and either $x \leq y$, $y \leq x$, or x and y are incomparable. In these cases, $P = x \oplus y$, $P = y \oplus x$, and $P = x + y$ respectively. Since these contradict the primality of P, such an x and y don't exist and all maximal RAPs are disjoint. Assume there exist two such partitions, then there must exist distinct RAPs S, T where $S \cap T \neq \emptyset$, but this violates maximality by the same argument as above. □

Finally we are ready to introduce the Ur-decomposition. We say a poset P is Ur-decomposable if $|P| = 1$ or if P can be expressed as $P = \mathscr{P}[x_k \to P_k]_{k=1}^n$, where $\mathscr{P}$ is a strong-prime, chain, or antichain and where each P_k is Ur-decomposable. In the case that $\mathscr{P}$ is a chain or antichain, we additionally insist that each P_k is maximal, in that it cannot be expressed as the result of an ordinal or direct sum respectively. Such a decomposition is called an Ur-decomposition.

Theorem 10. *All posets have a unique Ur-decomposition.*

Proof. It suffices to show that each nontrivial poset P can be uniquely expressed as $P = \mathscr{P}[x_k \to P_k]_{k=1}^n$ where $\mathscr{P}$ is a strong-prime, chain, or antichain.

Assume P is prime, let $\{P_k\}$ be the set of maximal RAPs which partition P, and let $\mathscr{P}$ be the poset defined on the RAPs. Assume $\mathscr{P}$ is not a strong prime, then by Proposition 2 $\mathscr{P}$ contains a non-trivial RAP S. Then the RAPs associated with S from an RAP in P, which violates the maximality of RAPs in our partition. Furthermore, P is prime, and thus cannot be decomposed into a chain or anti-chain, and the RAP partition of P is unique.

Assume P is not prime, then P is expressible as the result of a direct or ordinal sum, and existence of an Ur-decomposition as a chain or anti-chain is immediate. Furthermore, these options are exclusive, and the insistance on maximal chains gives uniqueness within each class. Then it must only be shown that such a P cannot be decomposed into a strong prime $\mathscr{P}$.

Let $P = \mathscr{P}[x_k \to P_k]_{k=1}^n$, because P is not prime, there exists a subposet S s.t. S has the same relation to every element in $P - S$. S must be contained at least partially in some P_i, but since P_i is a RAP it must have the same relation to every element in $P - P_i$. Furthermore, this implies $x_i \in \mathscr{P}$ has the same relation to $\mathscr{P} - x_i$. Then either $\mathscr{P} = A_2$ or C_2, or $|\mathscr{P}| > 2$ and must contain a non-trivial RAP. In either case, $\mathscr{P}$ cannot be a strong prime. □

4 Labeled Poset Recurrence

A labeling of a poset P is a bijective map, $\omega\colon P \to [|P|]$. Additionally, $\sigma\colon P \to [m]$ is a (P,ω)-partition of height m when σ satisfies

1. If $s < t$ in P and $\omega(s) < \omega(t)$, then $\sigma(s) \leq \sigma(t)$.
2. If $s < t$ in P and $\omega(s) > \omega(t)$, then $\sigma(s) < \sigma(t)$.

Finally, the number of (P,ω)-partitions of height m is denoted by $\Omega_{P,\omega}(m)$. For incomparable $x, y \in P$, let $P|x \leq y$ be the result of adding the cover relation $x \leq y$. This notation allows for the following recurrence.

Lemma 11. *For incomparable $x, y \in P$ and a labeling ω of P,*

$$\Omega_{P,\omega} = \Omega_{P|x\leq y,\omega} + \Omega_{P|y\leq x,\omega}.$$

Proof. Without loss of generality suppose that $\omega(x) < \omega(y)$. Let $\sigma\colon P \to [m]$ be a (P,ω)-partition. If $\sigma(x) \leq \sigma(y)$, then σ is only counted by $\Omega_{P|x\leq y,\omega}$. If $\sigma(y) < \sigma(x)$, then σ is only counted by $\Omega_{P|y\leq x,\omega}$. □

For a labeleing ω of a poset P, let $\overline{\omega} = m + 1 - \omega$ denote the dual labeling to ω.

Theorem 12 (Poset Reciprocity)**.** *For all labeled posets, (P,ω),*

$$\Omega_{P,\overline{\omega}}(m) = (-1)^{|P|}\Omega_{P,\omega}(-m).$$

Proof. We shall proceed by strong induction on the number of pairs of incomparable elements in P. For the base case where P has no pairs of incomparable elements, P is a chain. Then (P,ω) can be thought to be a chain with i strict edges and j non-strict edges where $i + j = |P| - 1$. Using a modified stars and bars technique, we get that $\Omega_{P,\omega}(m) = \binom{m+j}{|P|}$. Since $(P,\overline{\omega})$ is a chain with j strict edges and i non-strict edges, $\Omega_{P,\overline{\omega}}(m) = \binom{m+i}{|P|}$. Then by the binomial reciprocity theorem,

$$\Omega_{P,\overline{\omega}}(m) = \binom{m+j}{k} = (-1)^k\binom{-(m+j)+|P|-1}{|P|} = (-1)^k\binom{-|P|+i}{|P|} = (-1)^k\Omega_{P,\omega}(-m)$$

which shows the base case. Now suppose that the result holds for all posets with fewer than n pairs of incomparable elements and suppose that P has n pairs of incomparable elements. Then let $x, y \in P$ be incomparable. By Lemma 11 and our inductive assumption,

$$\begin{aligned}
\Omega_{P,\overline{\omega}}(m) &= \Omega_{P|x\leq y,\overline{\omega}}(m) + \Omega_{P|y\leq x,\overline{\omega}}(m)\\
&= (-1)^{|P|}\Omega_{P|x\leq y,\omega}(-m) + (-1)^{|P|}\Omega_{P|y\leq x,\omega}(-m)\\
&= (-1)^{|P|}\Omega_{P,\omega}(-m)
\end{aligned}$$

which shows the inductive step and completes the proof. □

If (P, ω) is a labeled poset, ω is a natural labeling when $\omega(s) < \omega(t)$ for all $s < t$ in P. It is well known that every poset has a natural labeling. Let $\overline{F}_P(m)$ denote the number of strict order-preserving maps $f\colon P \to [m]$. If ω is a natural labeling for P then $F_P = \Omega_{P,\omega}$ and $\overline{F}_P = \Omega_{P,\overline{\omega}}$

Corollary 13. *For all posets P,*

$$\overline{F}_P(m) = (-1)^{|P|} F_P(-m).$$

Proposition 14. *For all posets P, there exist $c_k \in \mathbb{N}$ such that*

$$F_P(m) = (-1)^{|P|} \sum_{k=h(P)}^{|P|} (-1)^k c_k \binom{m+k-1}{k}.$$

Proof. By Corollary 13 and the binomial reciprocity theorem, it suffices to show that there exist $c_k \in \mathbb{N}$ such that $\overline{F}_P(m) = \sum_{k=h(P)}^{|P|} c_k \binom{m}{k}$. Then let c_k be the number of surjective strict order-preserving maps $f\colon P \to [k]$. Since f is strict, f must assume $h(P)$ different values on a chain of height $h(P)$. Therefore $c_k = 0$ for $k < h(P)$. Since f is surjective, $c_k = 0$ for $k > |P|$. Then to count the number of strictly order preserving maps $f\colon P \to [k]$, iterate over the size of the image of f, k. For each k, there are $\binom{m}{k}$ ways to choose the image of f and c_k ways to choose f with the image of f specified. Then $\overline{F}_P(m) = \sum_k c_k \binom{m}{k}$. □

Lemma 15. *If $L\colon \mathbb{Q}[m] \to \mathbb{Q}[m]$ satisfies*

$$L\left(\binom{m}{c+d}\right) = L\left(\binom{m}{c}\right) L\left(\binom{m}{d}\right)$$

for all integer $c, d \geq 0$, then L also satisfies

$$L\left(\binom{m+a+b}{c+d}\right) = L\left(\binom{m+a}{c}\right) L\left(\binom{m+b}{d}\right)$$

for all integer $a, b, c, d \geq 0$.

Proof. We shall proceed by induction on $a+b$. For the base case where $a+b=0$, $a=b=0$ and the result is a re-indexed form of the assumption on L. Now suppose that the result holds for all $a+b=n$ and that $a+b=n+1$. Without loss of generality, $a \geq 1$. Then by Pascal's identity and the inductive assumption,

$$\begin{aligned}
L\left(\binom{m+a+b}{c+d}\right) &= L\left(\binom{m+(a-1)+b}{(c-1)+d}\right) + L\left(\binom{m+(a-1)+b}{c+d}\right) \\
&= L\left(\binom{m+(a-1)}{c-1}\right) L\left(\binom{m+b}{d}\right) + L\left(\binom{m+(a-1)}{c}\right) L\left(\binom{m+b}{d}\right) \\
&= L\left(\binom{m+(a-1)}{c-1} + \binom{m+(a-1)}{c}\right) L\left(\binom{m+b}{d}\right) \\
&= L\left(\binom{m+a}{c}\right) L\left(\binom{m+b}{d}\right)
\end{aligned}$$

which shows the inductive step and completes the proof. □

We can generalize $\oplus$ to labeled posets in the following way: Given labeled posets (P, ω) and (Q, ψ), let $\omega \oplus \psi$ be a labeling on $P \oplus Q$ given by

$$(\omega \oplus \psi)(x) = \begin{cases} \omega(x) & x \in P \\ |P| + \psi(x) & x \in Q \end{cases}.$$

Then $(P \oplus Q, \omega \oplus \psi)$ is the labeled poset where every element of P is weakly less than every element of Q.

Theorem 16. *For all labeled posets* $(P, \omega), (Q, \psi)$,

$$L(\Omega_{P\oplus Q,\omega\oplus\psi}) = L(\Omega_{P,\omega})L(\Omega_{Q,\psi})$$

where $L\colon \mathbb{Q}[m] \to \mathbb{Q}[m]$ *satisfies*

$$L\left(\!\!\binom{m}{c+d}\!\!\right) = L\left(\!\!\binom{m}{c}\!\!\right) L\left(\!\!\binom{m}{d}\!\!\right)$$

for all integer $c, d \geq 0$.

Proof. We shall proceed by strong induction on the number of pairs of incomparable elements in $P \oplus Q$. For the base case where $P \oplus Q$ has no pairs of incomparable elements, P and Q are chains. Suppose that (P, ω) has i strict edges and j non-strict edges and suppose that (Q, ψ) has k strict edges and l non-strict edges. Then $(P \oplus Q, \omega \oplus \psi)$ has $i + k$ strict edges and $j + l + 1$ non-strict edges. By the same computation of order polynomials of labeled chains as in Theorem 12, it suffices to show that

$$L\left(\!\!\binom{m+j+l+1}{i+j+k+l+2}\!\!\right) = L\left(\!\!\binom{m+j}{i+j+1}\!\!\right) L\left(\!\!\binom{m+k}{k+l+1}\!\!\right).$$

Then letting $a = j + 1$, $b = d + 1$, $c = i + j + 1$, $d = k + l + 1$ and applying Lemma 15 shows the base case. Now suppose that the result holds for all posets where $P \oplus Q$ has fewer than n pairs of incomparable elements and suppose that $P \oplus Q$ has n pairs of incomparable elements. Then without loss of generality, P has an incomparable pair of elements, x and y. Then by the linearity of L and our inductive assumption,

$$\begin{aligned}
L(\Omega_{P\oplus Q,\omega\oplus\psi}) &= L(\Omega_{(P\oplus Q)|x\leq y,\omega\oplus\psi} + \Omega_{(P\oplus Q)|y\leq x,\omega\oplus\psi}) \\
&= L(\Omega_{(P|x\leq y)\oplus Q,\omega\oplus\psi}) + L(\Omega_{(P|y\leq x)\oplus Q,\omega\oplus\psi}) \\
&= L(\Omega_{P|x\leq y,\omega})L(\Omega_{Q,\psi}) + L(\Omega_{P|y\leq x,\omega})L(\Omega_{Q,\psi}) \\
&= L(\Omega_{P|x\leq y,\omega} + \Omega_{P|y\leq x,\omega})L(\Omega_{Q,\psi}) \\
&= L(\Omega_{P,\omega})L(\Omega_{Q,\psi})
\end{aligned}$$

which shows the inductive step and completes the proof. $\square$

Corollary 17. *For all posets* P, Q,

$$L(F_{P\oplus Q}) = L(F_P)L(F_Q) \text{ and } L(\overline{F}P \oplus Q) = L(\overline{F}_P)L(\overline{F}_Q)$$

where $L\colon \mathbb{Q}[m] \to \mathbb{Q}[m]$ *satisfies*

$$L\left(\!\!\binom{m}{c+d}\!\!\right) = L\left(\!\!\binom{m}{c}\!\!\right) L\left(\!\!\binom{m}{d}\!\!\right)$$

for all integer $c, d \geq 0$.

Proof. Let ω and ψ be natural labelings of P and Q respectively. Then

$$L(F_{P\oplus Q}) = Ł(\Omega_{P\oplus Q,\omega\oplus\psi}) = L(\Omega_{P,\omega})L(\Omega_{Q,\psi}) = L(F_P)L(F_Q).$$

Additionally, by the poset reciprocity theorem,

$$L(\overline{F}_{P\oplus Q}(m)) = (-1)^{|P\oplus Q|}L(F_{P\oplus Q}(-m)) = (-1)^{|P|}L(F_P(-m))(-1)^{|Q|}L(F_Q(-m)) = L(\overline{F}_P(m))L(\overline{F}_Q(m)).$$

where m represents a variable (rather than a particular number). $\square$

Corollary 18. *For labeled posets* $(P, \omega), (P', \omega'), (Q, \psi), (Q', \psi')$, *any two conditions imply the third:*

1) $(P, \omega) \sim (P', \omega')$
2) $(Q, \psi) \sim (Q', \psi')$
3) $(P \oplus Q, \omega \oplus \psi) \sim (P' \oplus Q', \omega' \oplus \psi')$

Corollary 19. *For all labeled posets* $(P, \omega), (Q, \psi)$,

$$(P \oplus Q, \omega \oplus \psi) \sim (Q \oplus P, \psi \oplus \omega).$$

5 Order Polynomial Recurrence Relation

This section is mostly obsoleted by the previous section but is being kept for its recurrences and discussion of multivariate generating functions.

5.1 The Standard Recurrence

For all posets P and $m \in \mathbb{N}$, let $\overline{F}_P(m)$ denote the number of strict order-preserving maps $f\colon P \to [m]$. For incomparable $x, y \in P$, let $P|x \leq y$ be the result of adding the cover relation $x \leq y$ and let $P|x = y$ be the result of identifying x and y.

Lemma 20. *For incomparable $x, y \in P$,*

$$F_P = F_{P|x \leq y} + F_{P|y \leq x} - F_{P|x=y} \text{ and } \overline{F}_P = \overline{F}_{P|x \leq y} + \overline{F}_{P|y \leq x} + \overline{F}_{P|x=y}.$$

Proof. For any order preserving map $f\colon P \to [m]$, either $f(x) < f(y)$, $f(x) > f(y)$, or $f(x) = f(y)$. The first case is counted by $F_{P|x \leq y}$, the second case is counted by $F_{P|y \leq x}$, and the third case is counted by $F_{P|x \leq y}$, $F_{P|y \leq x}$, and $F_{P|x=y}$. Each case is counted exactly once. For any strict order preserving map $f\colon P \to [m]$, either $f(x) < f(y)$, $f(x) > f(y)$, or $f(x) = f(y)$. The first case is counted by $\overline{F}_{P|x \leq y}$, the second case is counted by $F_{P|y \leq x}$, and the third case is counted by $F_{P|x=y}$. Each case is counted exactly once. □

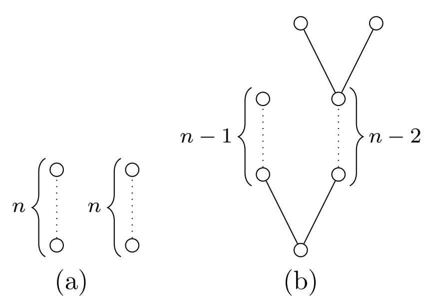


Figure 3: An infinite family of doppelgangers following from recurrence

Theorem 21 (Poset reciprocity)**.** *For all posets P, $\overline{F}_P(m) = (-1)^{|P|} F_P(-m)$.*

Proof. We shall proceed by strong induction on the number of pairs of incomparable elements in P. For the base case where P has no pairs of incomparable elements, P is a chain, $F_P(m) = \binom{m+|P|-1}{|P|}$ and $\overline{F}_P(m) = \binom{m}{|P|}$. Now suppose that the result holds for all posets with at most n pairs of incomparable elements and suppose that P has $n+1$ pairs of incomparable elements. Then by Lemma 3 and our inductive assumption,

$$\begin{aligned}
\overline{F}_P(m) &= \overline{F}_{P|x \leq y}(m) + \overline{F}_{P|y \leq x}(m) + \overline{F}_{P|x=y}(m) \\
&= (-1)^{|P|} F_{P|x \leq y}(-m) + (-1)^{|P|} F_{P|y \leq x}(-m) + (-1)^{|P|-1} F_{P|x=y}(-m) \\
&= (-1)^{|P|} (F_{P|x \leq y}(-m) + F_{P|y \leq x}(-m) - F_{P|x=y}(-m)) \\
&= (-1)^{|P|} F_P(-m)
\end{aligned}$$

which shows the inductive step and completes the proof. □

While the above only proves the reciprocity theorem for unlabeled posets, the proof is the same for the general form on (P, ω)-partitions.

Proposition 22. *For all posets P, there exist $c_k \in \mathbb{N}$ such that*

$$F_P(m) = (-1)^{|P|} \sum_{k=h(P)}^{|P|} (-1)^k c_k \binom{m+k-1}{k} \text{ and } \overline{F}_P(m) = \sum_{k=h(P)}^{|P|} c_k \binom{m}{k}.$$

Proof. By Theorem 12 and the identity $\binom{m+k-1}{k} = (-1)^k \binom{-m}{k}$, it suffices to show the second statement. Let c_k be the number of strictly order preserving surjective maps $f\colon P \to [k]$. Since f is strict, f must assume $h(P)$ different values on a chain of height $h(P)$. Therefore $c_k = 0$ for $k < h(P)$. Since f is surjective, $c_k = 0$ for $k > |P|$. Then to count the number of strictly order preserving maps $f\colon P \to [k]$, we shall iterate over k, the size of the image of f. For each k, there are $\binom{m}{k}$ ways to choose the image of f and c_k ways to choose f with the image of f specified. Then $\overline{F}_P(m) = \sum_k c_k \binom{m}{k}$. ☐

Lemma 23. *If $F_P = \sum_k a_k F_{C_k}$ and $\overline{F}_P = \sum_k b_k \overline{F}_{C_k}$ then*

$$F_{P\oplus Q} = \sum_k a_k F_{C_k \oplus Q},\ F_{Q\oplus P} = \sum_k a_k F_{Q\oplus C_k},\ \overline{F}_{P\oplus Q} = \sum_k b_k \overline{F}_{C_k\oplus Q},\ \overline{F}_{Q\oplus P} = \sum_k b_k \overline{F}_{Q\oplus C_k}.$$

Proof. We shall only prove the first statement as the proofs of the rest are analogous. We shall proceed by strong induction on the number of pairs of incomparable elements in P. For the base case where P has no pairs of incomparable elements, P is a chain and $F_P = F_{C_{|P|}}$. Now suppose that the result holds for all posets with at most n pairs of incomparable elements and suppose that P has $n+1$ pairs of incomparable elements. Then by Lemma 11 and our inductive assumption,

$$\begin{aligned} F_{P\oplus Q} &= F_{(P\oplus Q)|x\le y} + F_{(P\oplus Q)|y\le x} - F_{(P\oplus Q)|x=y} \\ &= F_{(P|x\le y)\oplus Q} + F_{(P|y\le x)\oplus Q} - F_{(P|x\le y)\oplus Q} \\ &= \sum_k c_k F_{C_k\oplus Q} + \sum_k d_k F_{C_k\oplus Q} - \sum_k e_k F_{C_k\oplus Q} \\ &= \sum_k (c_k + d_k + e_k) F_{C_k\oplus Q} \\ &= \sum_k a_k F_{C_k\oplus Q} \end{aligned}$$

where c_k, d_k, e_k are the coefficients of $F_{P|x\le y}$, $F_{P|y\le x}$, $F_{P|x=y}$ in the F_{C_k} basis. ☐

Proposition 24. *Let L, T be the linear maps such that $L(F_{C_k}(m)) = m^k$ and $T(\overline{F}_{C_k}(m)) = x^m$. Then $L(F_{P\oplus Q}) = L(F_P)L(F_Q)$, $T(F_{P\oplus Q}) = T(F_P)T(F_Q)$, $L(\overline{F}_{P\oplus Q}) = L(\overline{F}_P)L(\overline{F}_Q)$, $T(\overline{F}_{P\oplus Q}) = T(\overline{F}_P)T(\overline{F}_Q)$.*

Proof. Let $F_P = \sum_k a_k F_{C_k}$, $\overline{F}_P = \sum_k b_k \overline{F}_{C_k}$, $F_Q = \sum_k c_k F_{C_k}$, $\overline{F}_Q = \sum_k d_k \overline{F}_{C_k}$. By Lemma 15,

$$L(F_{P\oplus Q}) = L\left(\sum_j a_j F_{C_j\oplus Q}\right) = L\left(\sum_{j,k} a_j b_k F_{C_j\oplus C_k}\right) = \sum_{j,k} a_j b_k x^{j+k} = L\left(\sum_k a_k F_{C_k}\right) L\left(\sum_k b_k F_{C_k}\right)$$

which shows the first statement. The proof of the fourth statement is analogous. For the second, let x represent a real variable. Then by Lemma 11 and the first statement,

$$T(F_{P\oplus Q}(x)) = (-1)^{|P\oplus Q|} T(\overline{F}_{P\oplus Q}(-x)) = (-1)^{|P|+|Q|} T(\overline{F}_P(-x)) T(\overline{F}_Q(-x) = T(F_P(x))T(F_Q(x))$$

which shows the second statement. The proof of the third statement is analogous. ☐

Corollary 25. *For all posets P, Q, R, S, any two of the following relations guarantee the third:*
1) $P \sim Q$
2) $R \sim S$
3) $P \oplus R \sim Q \oplus S$.

Corollary 26. $P \oplus Q \sim Q \oplus P$

5.2 Improper Recurrences

Recall that labeled posets can be viewed as an assignement of strict and non-strict edges, where not all assignements are valid posets. For incomparable elements $x, y \in P$, let $P|x < y$ be the poset P with the added restriction $x < y$ and all implied restrictions. This restriction may create a contradicition in labeling, leading to a non-poset assignment–this is why we call the recurrences using this object improper. However, plane partitions, and thus the order polynomial and multivariate generating function, are always well defined on $P|x < y$. Thus we get the following two improper recurrences:

Lemma 27. *For incomparable $x, y \in P$,*

$$F_P = F_{P|x<y} + F_{P|y<x} + F_{P|x=y}$$
$$F_P = F_{P|x\leq y} + F_{P|y<x}$$

Proof. The proof is in the vein of lemma 11. For any order preserving map $f : P \to [m]$, either $f(x) < f(y)$, $f(x) > f(y)$, or $f(x) = f(y)$. In both of the above recurrences, each case is counted exactly once. □

Lemma 28. *The latter recurrence applies to the multivariate generating function as well:*

$$K_{(P,\omega)} = K_{(P|x<y,\omega)} + K_{(P|x\leq y,\omega)}$$

Proof. The multivariate generating function is a sum over all (P, ω)-partitions of functions depending only upon the partition itself, not the poset. Because $P|x \leq y$ and $P|y < x$ have exactly the same plane partitions as P, the sum of their generating functions is simply the sum of the same values of the function over all partitions of P, which is just $K_{(P,\omega)}$ □

McNamara-Ward offer four unexplained equivalences of size 5 as a spring board for further exploration. We note that the latter recurrence relation explains their first example.

Proposition 29. *The posets P, Q shown in figure 4 have equivalent multivariate generating functions.*

Proof. See figure 4 below for the decomposition.
For the above choices of x, y, we get $P|x < y = Q|x < y$ and $P|y \leq x = Q|y \leq x$. Then

$$K_{(P|x<y,\omega)} + K_{(P|y\leq x,\omega)} = K_{(Q|x<y,\omega)} + K_{(Q|y\leq x,\omega)}$$
$$K_{(P,\omega)} = K_{(Q,\omega)}$$

by Lemma 19 □

Because our work focuses on unlabeled posets, these improper recurrences are of little use; we offer them as an analog of the standard recurrence for multivariate generating functions.

6 Classification

The previous section introduces a highly restrictive invariant on doppleganger classes, height. Invariants which can be easily calculated allow for classification of doppelgangers of certain families of posets. The following lemma presents four such invariants that have simple recursive formulas over the operations of direct and ordinal sum.

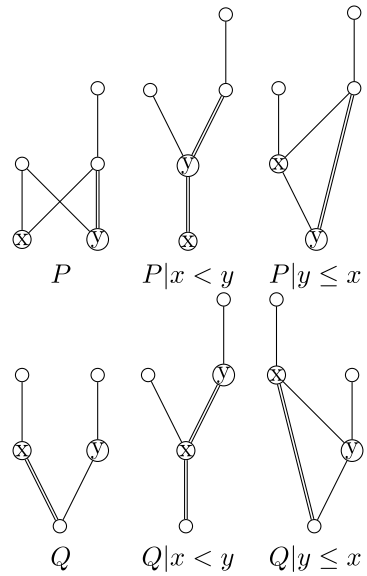


Figure 4: The latter strict recurrence

Lemma 30. *If $P \sim Q$ then $|P| = |Q|$, $F_P(2) = F_Q(2)$, $h(P) = h(Q)$, $l(P) = l(Q)$. Additionally,*

$$|P + Q| = |P| + |Q| \tag{1}$$
$$|P \oplus Q| = |P| + |Q| \tag{2}$$
$$F_{P+Q}(2) = F_P(2)F_Q(2) \tag{3}$$
$$F_{P\oplus Q}(2) = F_P(2) + F_Q(2) - 1 \tag{4}$$
$$h(P + Q) = \max(h(P), h(Q)) \tag{5}$$
$$h(P \oplus Q) = h(P) + h(Q) \tag{6}$$
$$l(P + Q) = \binom{|P| + |Q|}{|P|} l(P)l(Q) \tag{7}$$
$$l(P \oplus Q) = l(P)l(Q) \tag{8}$$

for all posets P, Q.

Proof. Note that $|P| = \deg F_P$, $h(P)$ is the index of the first nonzero term of F_P in the chain basis, and $l(P)$ is $(\deg F_P)!$ times the leading coefficient of F_P. Then all four invariants depend only on F_P which shows the first part of the lemma. (1) and (2) follow from the definitions. (3) follows from $F_{P+Q} = F_P F_Q$. For (4), to count the order preserving maps $f\colon P \oplus Q \to \{1, 2\}$, there are two cases. If f sends every element of P to 1 then there are $F_Q(2)$ ways to choose f on Q. If f sends some element of P to 2 then there are $F_P(2) - 1$ ways to choose f on P. Then there are $F_P(2) + F_Q(2) - 1$ ways to choose f on $P \oplus Q$. For (5), since all chains in P and Q are also chains in $P + Q$, we have that $h(P + Q) \geq \max(h(P), h(Q))$. Since each chain in $P + Q$ is either a chain in P or a chain in Q, we have that $h(P + Q) \leq \max(h(P), h(Q))$. For (6), since if C_1 and C_2 are chains in P and Q respectively then $C_1 \oplus C_2$ is a chain in $P \oplus Q$, we have that $h(P \oplus Q) \geq h(P) + h(Q)$. Since if C is a chain in $P \oplus Q$ then C can be expressed as $C_1 \oplus C_2$ where C_1 and C_2 are chains in P and Q respectively, we have that $h(P \oplus Q) \leq h(P) + h(Q)$. For (7), we will count the number of bijections $f\colon P + Q \to \{1, \ldots, |P| + |Q|\}$. There are $\binom{|P|+|Q|}{|P|}$ ways to choose $f(P)$ which determines $f(Q)$, there are $l(P)$ ways to choose f on P, and there are $l(Q)$ ways to choose f on Q. For (8), since every element of P is smaller than every element of Q, every bijective map $f\colon P \oplus Q \to \{1, \cdots, |P| + |Q|\}$ must send $P \to \{1, \cdots, |P|\}$ and $Q \to \{|P|+1, \cdots, |Q|\}$. Then there are $l(P)$ ways to choose f on P and $l(Q)$ ways to choose f on Q. □

Lemma 31. *For any poset P,*

$$F_P(x) = \frac{(x+h(P)-1)!l(P)}{(x-1)!|P|!} \prod_{k=1}^{|P|-h(P)} (x+c_k) \text{ where } \prod_{k=1}^{|P|-h(P)} (1+c_k) = \frac{|P|!}{h(P)!l(P)}$$

for some $c_k \in \mathbb{C}$.

Proof. Since P has height $h(P)$, F_P has the roots $0, -1, \cdots, -h(P)+1$. Let $-c_1, \cdots, -c_{|P|-h(P)}$ be the remaining roots of F_P. Then since the leading coefficient of F_P is equal to $l(P)/|P|!$, we have that

$$F_P(x) = \frac{l(P)}{|P|!} \prod_{k=0}^{h(P)-1} (x+k) \prod_{k=1}^{|P|-h(P)} (x+c_k) = \frac{(x+h(P)-1)!l(P)}{(x-1)!|P|!} \prod_{k=1}^{|P|-h(P)} (x+c_k).$$

The condition on the product of c_k follows from setting $x = 1$. □

Proposition 32. *If $h(P) = |P| - 1$ then $P \sim Q$ if and only if $|P| = |Q|$, $h(P) = h(Q)$, and $l(P) = l(Q)$.*

Proof. Lemma 6 shows the forward direction. Now suppose that $|P| = |Q|$, $h(P) = h(Q)$, $l(P) = l(Q)$. By Lemma 5,

$$F_P(x) = \frac{(x+h(P)-1)!l(P)}{(x-1)!|P|!} \left(x + \frac{|P|}{l(P)} - 1\right) = \frac{(x+h(Q))!l(Q)}{x!|Q|!} \left(x + \frac{|Q|}{l(Q)} - 1\right) = F_Q(x)$$

which shows that $P \sim Q$. □

Proposition 33. *If $h(P) = |P|-2$ then $P \sim Q$ iff $|P| = |Q|$, $F_P(2) = F_Q(2)$, $h(P) = h(Q)$, and $l(P) = l(Q)$.*

Proof. Lemma 6 shows the forward direction. Now suppose that $|P| = |Q| = n$, $F_P(1) = F_Q(1)$, $h(P) = h(Q) = h$, $l(P) = l(Q) = l$. By Lemma 5 there exist constants $c, d \in \mathbb{C}$ such that

$$F_P(x) = \frac{(x+h-1)!l}{(x-1)!n!}(x+c-1)\left(x + \frac{n(n-1)}{cl} - 1\right)$$

and

$$F_Q(x) = \frac{(x+h-1)!l}{(x-1)!n!}(x+d-1)\left(x + \frac{n(n-1)}{dl} - 1\right).$$

Setting $F_P(2) = F_Q(2)$ gives that

$$(c+1)\left(1 + \frac{n(n-1)}{cl}\right) = (d+1)\left(1 + \frac{n(n-1)}{dl}\right).$$

Multiplying both sides by cd, expanding, and simplifying gives that

$$c^2 d + \frac{n(n-1)}{l} d = cd^2 + \frac{n(n-1)}{l} c.$$

Rearranging terms gives that

$$cd(c-d) = \frac{n(n-1)}{l}(c-d).$$

Then either $c = d$ or $c = \frac{n(n-1)}{d}$ and in either case $F_P = F_Q$. □

Infinite Family	$l(P)$	$F_P(2)$
$Tri(m_1, m_2, m_3)$	$m_2 + 1$	$\lvert P\rvert + m_2 + 1$
$Dtri(m_1, m_2, m_3, m_4, m_5)$	$(m_2 + 1)(m_4 + 1)$	$\lvert P\rvert + m_2 + m_4 + 1$
$Ntri(m_1, m_2, m_3, m_4, m_5)$	$(m_2 + m_3 + m_4 + 2)(m_3 + 1)$	$\lvert P\rvert + m_2 + 3m_3 + m_4 + 2$
$Xdis(m_1, m_2, m_3, m_4, m_5)$	$(m_2 + m_3 + 1)(m_3 + m_4 + 1) + m_3 + 1$	$\lvert P\rvert + m_2 + 3m_3 + m_4 + 2$
$Xcon(m_1, m_2, m_3, m_4, m_5)$	$(m_2 + m_3 + 1)(m_3 + m_4 + 1) - \frac{1}{2}m_3(m_3 + 1)$	$\lvert P\rvert + m_2 + 2m_3 + m_4 + 1$

Table 1: Values of $l(P)$ and $F_P(2)$ for the five infinite families in Figure 1.

These results suggest a new way to classify all doppelgangers of posets with larger height. In particular, if we can compute $l(P)$ and $F_P(2)$ for all posets with $|P| = n$, $h(P) = n - 2$, we would be able to find all doppelgangers of such posets. In the next couple results, we prove that such posets fall into only a few infinite families.

Lemma 34. *If $x_1 \le \cdots \le x_n$ is a chain in P and x is some other element of P, then there exist nonnegative integers $m_1+m_2+m_3 = n$ such that x is greater than $x_1, \cdots, x_{m_1}$, x is incomparable to $x_{m_1+1}, \cdots, x_{m_1+m_2}$, and x is less than $x_{m_1+m_2+1}, \cdots, x_{m_1+m_2+m_3}$.*

Proof. Let m_1 be maximal such that $x_{m_1} \le x$, let m_2 be minimal such that $x \le x_{m_1+m_2+1}$, and let $m_3 = n-m_1-m_2$. Then by transitivity, x is greater than $x_1, \cdots, x_{m_1}$ and x is less than $x_{m_1+m_2+1}, \cdots, x_{m_1+m_2+m_3}$. Additionally, x is neither less than nor greater than $x_{m_1+1}, \cdots, x_{m_1+m_2}$. □

Proposition 35. *All posets P with $|P| - h(P) = 1$ are isomorphic to a poset depicted by Figure 1(a).*

Proof. Let C be a maximal chain in P and let x be the remaining element of P. Let m_1, m_2, m_3 be the result of applying lemma 4 to C and x. Then $P \cong Tri(m_1, m_2, m_3)$. □

Proposition 36. *All posets P with $|P| - h(P) = 2$ are isomorphic to poset depicted by Figures 1(b-h).*

Proof. Let C be a maximal chain in P and let x, y be the two remaining elements of P. Let m_1, m_2, m_3 be the result of applying lemma 4 to C and x and let n_1, n_2, n_3 be the result of applying lemma 4 to C and y. Then

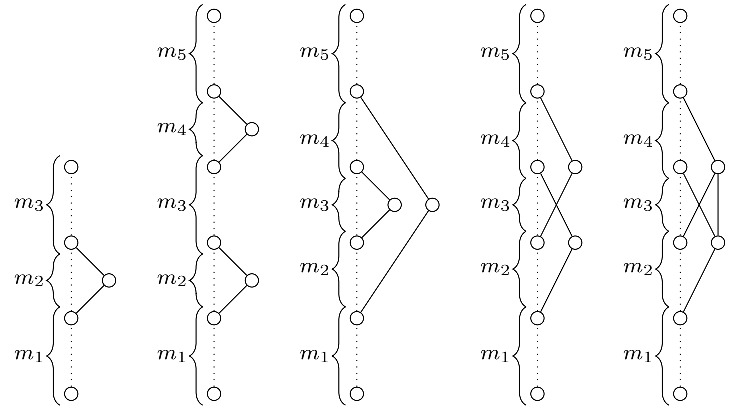
$$P \cong \begin{cases} Ntri(m_1, n_1 - m_1, n_2, n_3 - m_3, m_3) & m_1 \le n_1,\ m_3 \le n_3,\ x \text{ and } y \text{ incomparable} \\ Ntri(n_1, m_1 - n_1, m_2, m_3 - n_3, n_3) & m_1 \ge n_1, m_3 \ge n_3,\ x \text{ and } y \text{ incomparable} \\ Xdis(m_1, n_1 - m_1, m_1 + m_2 - n_1, m_3 - n_3, n_3) & m_1 \le n_1, m_3 \ge n_3,\ x \text{ and } y \text{ incomparable} \\ Xdis(n_1, m_1 - n_1, n_1 + n_2 - m_1, n_3 - m_3, m_3) & m_1 \ge n_1, m_3 \le n_3,\ x \text{ and } y \text{ incomparable} \\ Xcon(m_1, n_1 - m_1, m_1 + m_2 - n_1, m_3 - n_3, n_3) & m_1 + m_2 - n_1 \ge 0,\ x \le y \\ Xcon(n_1, m_1 - n_1, n_1 + n_2 - m_1, n_3 - m_3, m_3) & n_1 + n_2 - m_1 \ge 0,\ y \le x \\ Dtri(m_1, m_2, n_1 - m_1 - m_2, n_2, n_3) & n_1 - m_1 - m_2 \ge 0,\ x \le y \\ Dtri(n_1, n_2, m_1 - n_1 - n_2, m_2, m_3) & m_1 - n_1 - n_2 \ge 0,\ y \le x \end{cases}.$$

□

Proposition 37. *The values of $l(P)$ and $F_P(2)$ of the posets depicted in Figure 1 are given by Table 1.*

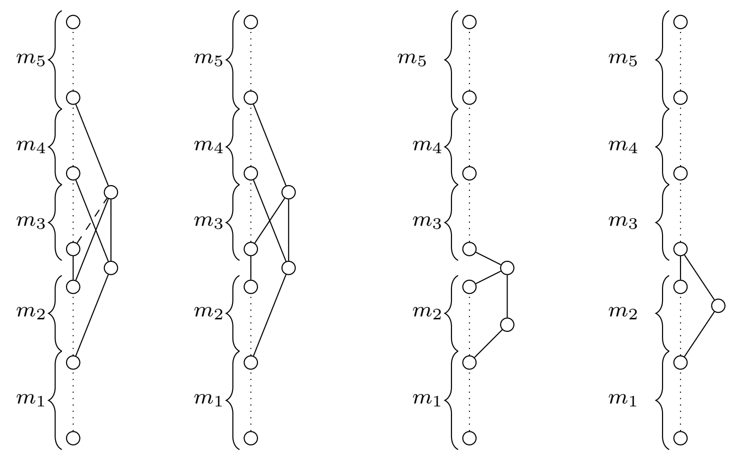
Proof. Note that

$$\begin{aligned} Tri(m_1, m_2, m_3) &\cong C_{m_1} \oplus (C_{m_2} + C_1) \oplus C_{m_3} \\ Dtri(m_1, m_2, m_3, m_4, m_5) &\cong C_{m_1} \oplus (C_{m_2} + C_1) \oplus C_{m_4} \oplus (C_{m_3} + C_1) \oplus C_{m_5} \\ Ntri(m_1, m_2, m_3, m_4, m_5) &\cong C_{m_1} \oplus ((C_{m_2} \oplus (C_{m_3} + C_1) \oplus C_{m_4}) + C_1) \oplus C_{m_5}. \end{aligned}$$



(a) Tri (b) $Dtri$ (c) $Ntri$ (d) $Xdis$ (e) $Xcon$

Figure 5: Five infinite families of posets.



(a) $Xcon$ (b) $Xcon|x \leq y$ (c) $Xcon|y \leq x$ (d) $Xcon|x = y$

Figure 6: Chain decomposition of $Xcon$

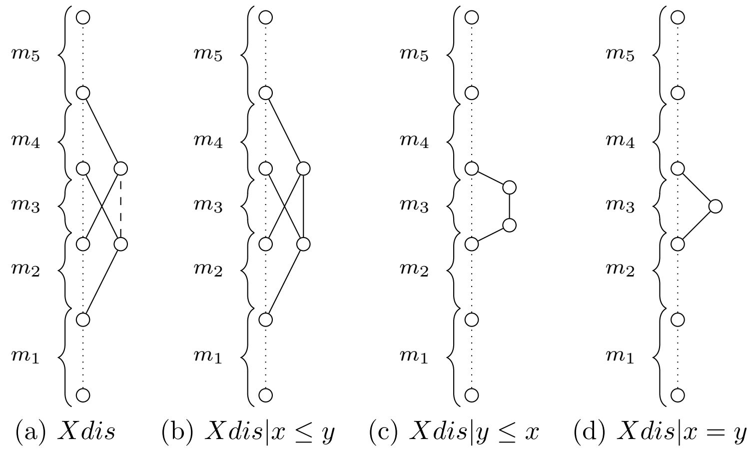


Figure 7: Chain decomposition of $Xdis$

Then the formulas for Tri, $Dtri$, and $Ntri$ follow from Lemma 3. For $Xcon$, note that by chain decomposition, Figure 2, and the formula for $l(Dtri)$,

$$\begin{aligned} l(Xcon(m_1, m_2, m_3, m_4, m_5)) &= l(Xcon(m_1, m_2+1, m_3-1, m_4, m_5)) + l(Tri(m_1, m_2, m_3+m_4+m_5+1)) \\ &= l(Xcon(m_1, m_2+1, m_3-1, m_4, m_5)) + m_2 + 1 \end{aligned}$$

and by induction,

$$\begin{aligned} l(Xcon(m_1, m_2, m_3, m_4, m_5)) &= l(Xcon(m_1, m_2+m_3, 0, m_4, m_5)) + \sum_{k=1}^{m_3}(m_2+k) \\ &= l(Dtri(m_1, m_2+m_3, 0, m_4, m_5)) + m_2 m_3 + \frac{1}{2}m_3(m_3+1) \\ &= (m_2+m_3+1)(m_3+m_4+1) - \frac{1}{2}m_3(m_3+1). \end{aligned}$$

Again, by chain decomposition, Figure 2, and the formula for $F_{Dtri}(2)$,

$$\begin{aligned} F_{Xcon(m_1,m_2,m_3,m_4,m_5)}(2) =& F_{Xcon(m_1,m_2+1,m_3-1,m_4,m_5)}(2) + F_{Tri(m_1,m_2,m_3+m_4+m_5+1)}(2) \\ &- F_{Tri(m_1,m_2,m_3+m_4+m_5)} \\ =& F_{Xcon(m_1,m_2+1,m_3-1,m_4,m_5)}(2) + 1 \end{aligned}$$

and by induction,

$$\begin{aligned} F_{Xcon(m_1,m_2,m_3,m_4,m_5)}(2) &= F_{Xcon(m_1,m_2+m_3,0,m_4,m_5)}(2) + m_3 \\ &= F_{Dtri(m_1,m_2+m_3,0,m_4,m_5)}(2) + m_3 \\ &= |P| + m_2 + 2m_3 + m_4 + 1. \end{aligned}$$

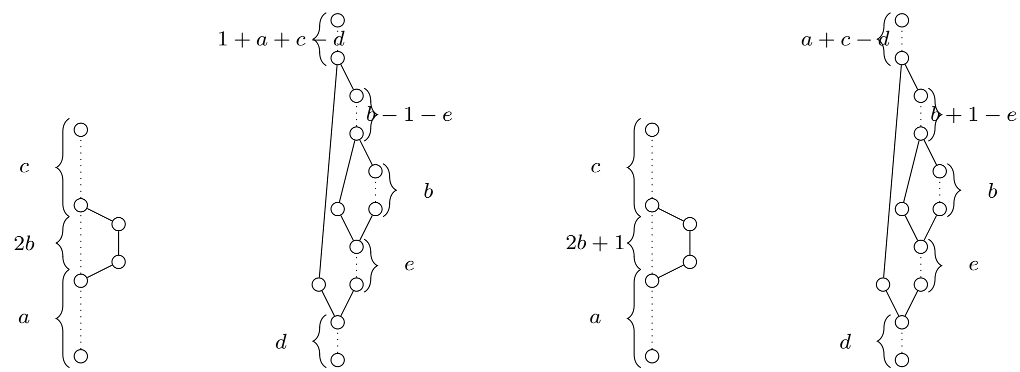
For $Xdis$, note that by chain decomposition, Figure 3, and the formula for $l(Xcon)$,

$$\begin{aligned} l(Xdis(m_1, m_2, m_3, m_4, m_5)) &= l(Xcon(m_1, m_2, m_3, m_4, m_5)) + l(Xcon(m_1+m_2, 0, m_3, 0, m_4+m_5)) \\ &= (m_2+m_3+1)(m_3+m_4+1) + (m_3+1)(m_3+1) - m_3(m_3+1) \\ &= (m_2+m_3+1)(m_3+m_4+1) + m_3 + 1. \end{aligned}$$

Again, by chain decomposition, Figure 3, and the formulas for $F_{Xcon}(2)$ and $F_{Tri}(2)$,

$$\begin{aligned} F_{Xdis(m_1,m_2,m_3,m_4,m_5)}(2) =& F_{Xcon(m_1,m_2,m_3,m_4,m_5)} + F_{Xcon(m_1+m_2,0,m_3,0,m_4+m_5)} \\ & - F_{Tri(m_1+m_2,m_3,m_4+m_5)} \\ &= |P| + m_2 + 2m_3 + m_4 + 1 + |P| + 2m_3 + 1 - (|P| - 1 + m_3 + 1) \\ &= |P| + m_2 + 3m_3 + m_4 + 2. \end{aligned}$$

□



(a) $X^c(a, 0, 2b, 0, c)$ (b) (c) $X^c(a, 0, 2b + 1, 0, c)$ (d)

Figure 8: Two Infinite Families of Doppelgangers which follow from the above

7 Further directions

7.1 Roots of the Order Polynomial

Throughout the paper we have proved and put to use certain properties of the roots of the order polynomial. Most importantly, given a poset P, we found that the only integer roots of $F_P(m)$ are $0, -1, \cdots, -h(P)+1$. It is also the case that because $F_P(1) = 1$ the product of the roots of $F_P(m-1)$ must be $\frac{|P|!}{e(P)}$, the reciprocal of the leading coefficient. We believe that this is only a small portion of the structure of order polynomials. Experimentally, simply plotting the roots of order polynomials of a large number of posets gives the following conjectures

Conjecture 38. *For any poset P and root of the order polynomial x, $Re(x) < 1$*

Conjecture 39. *For any poset P and root of the order polynomial x, $|x| < M$, where $M \in \mathbb{R}^+$*

In fact, for small $|P|$, it appears that $M < |P|$. Whether or not this in particular holds for large posets, M is likely to be small.

Plotting the roots of the order polynomial for specified heights leads to another conjecture

Conjecture 40. *For any poset P with $h(P) = 2$, the roots of $h(P)$ are symmetric in the complex plane about $x = -\frac{1}{2}$*

Of course, the roots are also symmetric about the x-axis, that is to say if v is a root then so is $\bar{v}$. This is the case for all order polynomials. The above conjecture could be a step towards classifying posets of height 2, as knowing even a single root will often specify up to three others.

7.2 Families Closed Under Chain Decomposition

We observed an interesting phenomenon while classifying posets of height $|P| - 2$. Notably, we were able to use the fact that each family decomposed under a single step of chain decomposition into itself or simple posets. This allowed us to calculate invariants of entire families (and in that case to classify them), but the same method could have been used to calculate the order polynomial of each family. Such families can be thought of as closed under chain decomposition. Studying such families would not lead to further classification unless more invariants of doppelgangers are discovered, but it could be used to find previously unknown simple infinite families. While it is unclear at the moment how many such families exist, having happened upon one at random it seems likely that many others exist, For instance, series-parallel posets may always be decomposed into series-parallel posets, though the order polynomials of such are easy to calculate making this family of less interest in this case.

7.3 Single Step Chain Decomposition

Recall that we were able to use a single step of chain decomposition to find an infinite family of doppelgangers for C_n+C_n. What other dopplegangers can be explained through a single set of chain decomposition? C_n+C_n has high structural symmetry and simplicity, can the dopplegangers which cannot be explained by a single step of chain decomposition be classified?

7.4 Ur-Equivalence and Commutativity

Chain Decomposition proved that permuting the sequence of posets corresponding to a chain or anti-chain in the Ur-Decomposition preserves the order polynomial. We call posets where every permutation is allowed commutative, likewise when only two points may be permuted we call these points commutative.

Definition 41. *Elements $x, y \in P$ are Ur-commutative if $P[x \to Q] \sim P[y \to Q]$ for all posets Q. Similarly P is Ur-commutative if $P[x \to Q] \sim P[y \to Q]$ $\forall$ $x, y \in P$*

Definition 42. *Elements $x \in P$ and $y \in Q$ are Ur-equivalent if $P[x \to M] \sim Q[y \to M]$ for all posets M. Similarly P is Ur-equivalent to Q if $\forall$ $x \in M$ $\exists$ $y \in Q$ s.t. $P[x \to M] \sim P[y \to M]$ $\forall$ $x, y \in P$*

Note that element-pair Ur-equivalence is a generalization of element-pair commutativity with $P = Q$, and P is always Ur-equivalent to itself.

In the case of chains as above, commutativity is intuitively clear due to the symmetry of such posets, but it is natural to examine the general case. A deeper understanding of the order polynomial under Ur-Operations is necessary for classifying commutativity of elements or posets. We offer one possible tool equivalent to commutativity and which suggests two sufficient and necessary conditions

Definition 43. *Given any poset P and $x \in P$, define $F_P(x, m)_i$ to be the number of plane partitions $f : P - \{x\} \to [1, m]$ such that*

$$1 + \min_{x \leq y} f(y) - \max_{y \leq x} f(y) = i$$

Proposition 44. $\sum_{i=1}^{m} iF_P(x.m)_i = F_P(m)$ *and* $\sum_{i=1}^{m} F_P(x.m)_i = F_{P-\{x\}}(m)$

Proof. Recall that $F_P(x.m)_i$ denotes the number of plane partitions on $P - \{x\}$ such that $\min_{x \leq y} f(y) - \max_{y \leq x} f(y) = i - 1$. The possible differences range from 0 to $m - 1$ and for each partition with a difference of $i - 1$, x can take on i values. Similarly, simply summing $F_P(x, m)_i$ counts all plane partitions of $P - \{x\}$. □

Definition 45. *Given any poset P and $x \in P$, define*

$$(M_{x \in P})_{ij} = \begin{cases} F_P(x, j)_i & \text{if } j \geq i, \\ 0 & \text{if } j < i. \end{cases}$$

where $1 \leq i, j \leq |P| + 1$

Note that this matrix is simply a way of storing all requisite vector information, there is no additional structure in the matrix itself. However, the matrix is convenient as it is equivalent to commutativity.

Proposition 46. *$x, y \in P$ commute if and only if $M_{x\in P} = M_{y\in P}$*

Proof. The backward direction follows immediately from the definition of the matrix and lemma 1. If x, y commute, then we have $\sum_{i=1}^{m+1} F_P(x,m)_i * F_Q(i) = \sum_{i=1}^{m+1} F_P(y,m)_i * F_Q(i)$ for all posets Q. Consider using the sequence of posets $A_1 ... A_{m+1}$. Let $F_P(x,m)_i - F_P(y,m)_i = c_i$. This gives the system of equations

$$\sum_{i=1}^{m+1} i * c_i = 0$$

$$\vdots$$

$$\sum_{i=1}^{m} i^{m+1} * c_i = 0$$

Note the vectors $(1^n, ..., m^n)$ are linearly independent for $0 \leq n \leq m$, but then the system has a unique solution. We already know $c_i = 0$ is a solution, thus we have equality for all $F_P(x,m)_i$ and $F_P(y,m)_i$ which implies the result. □

Furthermore, equality of the matrix combined with Proposition 5 immediately gives the following result

Corollary 47. *If $x \in P$ and $y \in Q$ are Ur-equivalent, then $P \sim Q$ and $P - \{x\} \sim Q - \{y\}$*

Then the conditions above are necessary, which leads to the following conjecture.

Conjecture 48. *$x \in P$ and $y \in Q$ are Ur-equivalent if and only if $P \sim Q$ and $P - \{x\} \sim Q - \{y\}$*

Note that this conjecture would provide necessary and sufficient conditions both for Ur-equivalence and Ur-commutativity, as the latter is a specification of the former. This result holds posets of a small size, and in general for $P = Q$, $P - \{x\} = Q - \{y\}$. However, without a deeper understanding of the order polynomial under the Ur-operation, there is little hope of proving the conjecture true. If one could calculate the order polyonimal of large, non series-parallel posets, there would be some hope of finding a counter example as well.

8 Acknowledgements

We would like to thank Professor Morrow, head of the University of Washington REU, for making this research a reality for us. Likewise this research would not have been possible without the guidance of our mentor, Dr. Hamaker, who presented the problem to us initially and provided direction for our research, and Sean Griffin, who was especially helpful in situating our research within the literature. Lastly we would like to thank Christoph Kinzel for his insightful contributions to our early understanding of doppelgangers.

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